Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 10, 11

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Textbook Sections: 7.2, 7.3

Todays Topic: Trig substitution (and more of monday, if necessary)

In todays lecture, we will take on some more complicated integrals using trig identities. In a sense, though, what we do today will be a sort of "reverse version" of what we did on monday.

Our Focus today is on integrals involving terms like

$$\sqrt{a^2-x^2}\,, \quad \sqrt{a^2+x^2}\,, \quad \sqrt{x^2-a^2} \qquad (a>0).$$

which tend to show up — a **lot**.

Guiding Example:

Consider the integral $\int \sqrt{a^2-x^2}\,dx$.

① Let's do a new sort of change of variables. The new variable will be θ, and it will depend on the original variable x via the equation

$$\begin{cases} x = a\sin\theta \\ dx = a\cos\theta\, d\theta \end{cases} \quad \text{or} \quad \begin{cases} x = a\cos\theta. \\ dx = -a\sin\theta\, d\theta \end{cases}$$

② Introducing this change to the integral, we get (e.g.)

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$$

(3) Now, using the Pythagorean identity we can eliminate the root:

$$\int \sqrt{a^2-x^2}\,dx = \int a\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta = \int a\cos^2\theta\,d\theta = \cdots$$

⚠ We can make a substitution like this provided that the equation $x=f(\theta)$ (in general) uniquely defines θ as a function of x. That is, θ must lie in an interval where $f(\cdot)$ is a one-to-one (injective) function. (this means that $f(\theta)=f(\xi)$ iff $\theta=\xi$).

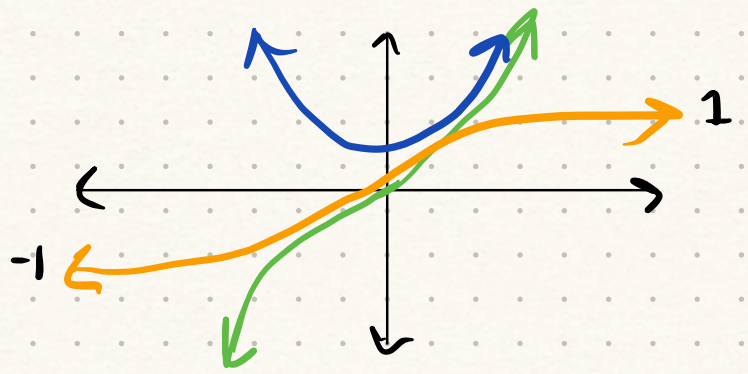
⚠ SO: in $x=a\sin(\theta)$, we need make a restriction like $\theta\in[-\pi/2,\pi/2]$.

Here are the substitutions:

Expression	Identity to leverage	Substitution
$\sqrt{a^2-x^2}$	$\sin^2\theta+\cos^2\theta=1$	$x=a\sin\theta$, $x=a\cos\theta$ $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$
$\sqrt{a^2+x^2}$	• $1+\tan^2\theta=\sec^2\theta$ or • $\cosh^2 t-\sinh^2 t=1$	• $x=a\tan\theta$ $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ or • $x=a\sinh t$ $t\in(-\infty,\infty)$
$\sqrt{x^2-a^2}$	• $1+\tan^2\theta=\sec^2\theta$ or • $\cosh^2 t-\sinh^2 t=1$	• $x=a\sec\theta$ $\theta\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$ or • $x=a\cosh t$ $t\in(0,\infty)$

⚠ If you aren't familiar with the hyperbolic trig functions

$$\begin{cases} \cosh x = \dfrac{e^x + e^{-x}}{2} \\ \sinh x = \dfrac{e^x - e^{-x}}{2} \\ \tanh x = \sinh x / \cosh x \end{cases}$$



... Don't worry! They aren't any more complicated than sine, cosine, or tangent, and they satisfy many analogous identities. In any case, we <u>don't</u> need to use them for this task — it's just good to know that they can handle some problems quite efficiently.

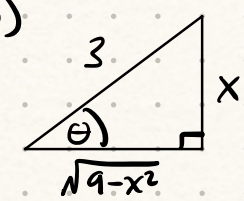
⚠ After carrying out the trig-substitution, we obtain a trig integral — if we do the indefinite integral we then need to switch variables **back to x**. To do this, it will be commonplace to "use geometry" to convert expressions in θ to expressions in x. It's best to see how this works in <u>examples</u>.

Examples:

① $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cdot\cos\theta\,d\theta = \int \cot^2\theta = \int (\csc^2\theta - 1) = -\cot\theta - \theta + C$

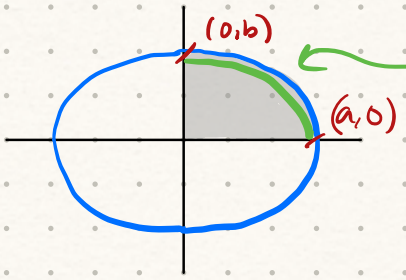
$x = 3\sin\theta$
$\theta \in [-\pi/2, \pi/2]$ (so $\cos \geq 0$)

To convert back to x:



$\rightsquigarrow \cot\theta = \dfrac{\sqrt{9-x^2}}{x}$, $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

(2) Compute the area enclosed by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the xy plane.



graph of $f(x)=\frac{b}{a}\sqrt{a^2-x^2}$

$$\frac{1}{4}A=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\int_0^{\pi/2}\frac{b}{a}(a\cos\theta)(a\cos\theta)\,d\theta$$

$x=a\sin\theta$, $\theta\in[0,\frac{\pi}{2}]$

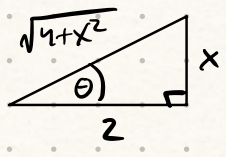
$$=ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=ab\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=\frac{\pi ab}{4}.$$

(3) $$\int\frac{1}{x^2\sqrt{x^2+4}}\,dx=\int\frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\cdot 2\sec\theta}=\frac{1}{4}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta=\frac{1}{4}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$x=2\tan x$, $x\in(-\frac{\pi}{2},\frac{\pi}{2})$



$$=-\frac{1}{4}\frac{1}{\sin\theta}+C$$

$$=-\frac{1}{4}\frac{\sqrt{4+x^2}}{x^2}+C$$

(4) $\int\frac{x}{\sqrt{x^2+n}}\,dx$. Do a u-sub instead!

(5) $\int\frac{dx}{\sqrt{x^2-a^2}}$, $a>0$. Homework 4: Do with sec(x) and cosh(x).

(6) $$\int_0^3\frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

1) First, lets set $x=\frac{3}{2}\tan\theta$, $dx=\frac{3}{2}\sec^2\theta\,d\theta$, which gives

$$4x^2+9=4(\tfrac{3}{2})^2\tan^2\theta+9=9(\tan^2\theta+1)=9\sec^2\theta.$$

Thus

2) $$\int_0^{3\sqrt{3}/2}\frac{x^3}{(4x^2+9)^{3/2}}\,dx=\int_0^{\frac{\pi}{3}}\frac{(\frac{3}{2})^3\tan^3x\cdot\frac{3}{2}\sec^2\theta}{(9\sec^2\theta)^{3/2}}\,d\theta=\frac{3}{16}\int_0^{\frac{\pi}{3}}\frac{\tan^3\theta}{\sec\theta}\,d\theta$$

(since $\frac{3}{2}\tan 0=0$, $\frac{3}{2}\tan(\frac{\pi}{3})=\frac{3}{2}\cdot\sqrt{3}$)

3) Now we handle this trig integral:

$$\frac{3}{16}\int_0^{\frac{\pi}{3}}\frac{\tan^3\theta}{\sec\theta}\,d\theta=\frac{3}{16}\int_0^{\frac{\pi}{3}}\frac{\sin^3\theta}{\cos^2\theta}\,d\theta=\frac{3}{16}\int_0^{\frac{\pi}{3}}\frac{\sin^2\theta}{\cos^2\theta}\sin\theta\,d\theta$$

$$=\frac{3}{16}\int_0^{\frac{\pi}{3}}\frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\,d\theta$$

u-sub: $u=\cos\theta$, $du=-\sin\theta\,d\theta$

$$=-\frac{3}{16}\int_1^{\frac{\sqrt{3}}{2}}\frac{1-u^2}{u^2}\,du$$